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Glassy properties of frustrated arrays of nonlinear devices

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We study linear arrays of different number of quartic oscillators shaped in the form of a ring when Gaussian noise (temperature) is added. Frustration is introduced through periodic boundary conditions and repulsive, directional interactions between neighboring oscillators. We show that these systems have similar dynamic properties than the arrays of fluxgates magnetometers. We find that there is a critical number of oscillators separating the regimes arising for systems with few and many oscillators and show that they reach an optimum ordering for a nonvanishing temperature. We also find that they have a relaxation process with an infinite mean life that is typical of glassy systems.

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I. INTRODUCTION

The behavior of open dynamical systems in the presence of external noise is a vast field of research. When noise is weak the dynamics of the system prevails and a purely deterministic behavior is observed. In the opposite extreme, no trace is left of the dynamical structure of the system. In an intermediate region in which neither picture is dominant, noise can play a constructive role. A well known case is that of stochastic resonance (SR) [1,2] in which noise magnifies a periodic external driving force switching a bistable system between its possible equilibrium positions.

A second widely studied case is that of Brownian motors [3] in which noise makes it possible to profit a broken left-right symmetry of a periodic potential to allow a net transport of particles along a ratchet. In Ref. [4] we have studied systems consisting in many elastically coupled particles in a periodic, externally driven, ratchet potential. It was then found that a variety of orbits pertaining to the deterministic dynamics can be recovered by the addition of external noise.

The constructive effect of noise has also been studied in numerical simulations [5] of large arrays of bistable devices in a regime of stochastic resonance showing an overall collective enhancement of the effects of an external driving force. Along this same line of research it was found that noise sustains the transmission of a periodic signal driving the first link of a linear array of oscillators in which each one feeds the next through a directional coupling, provided that all devices are in the regime of SR [6].

In [7] we have considered linear arrays that are closed in the form of a ring. It was found that a traveling wave can be sustained by noise. In Refs. [8,9] a similar ring involving fluxgates magnetometers has been studied with the important modification that the interaction between neighboring devices was taken to be repulsive. In this case the combined effect of noise and frustration produces spatiotemporal patterns that undergo a complex relaxation process.

The purpose of the present paper is to characterize the dynamical response of frustrated system as the ones reported in Ref. [9] consisting in either few or many nonlinear (quartic) bistable oscillators that are arranged in the form of a ring and coupled through a directional, repulsive, effective interaction by which the amplitude of one oscillator drives the next. Although the quartic potential that we have considered is different from the one of a fluxgate magnetometer [8] it nevertheless gives rise to a completely similar dynamic behavior. We take this opportunity to provide a quantitative study of the relaxation process reported in Ref. [9] and also to provide an analysis of a simple array of few nonlinear devices from the point of view of dynamical systems that helps to understand many of the features of these systems.

In these systems frustration can give rise to a dynamical response consisting in a solitonlike traveling wave along the ring [9]. We show that it corresponds to a global bifurcation that remains associated to the collapse of saddle-node pairs and that this deterministic dynamics can be recovered by the addition of noise as in Ref. [4] mentioned above. We also show that there are additional stable roots that proliferate with a growing number of links of the ring giving rise to the well known landscape with many equivalent equilibrium configurations that are typical of statistical frustrated systems.

The above-mentioned traveling wave consists in the switching of successive bistable devices from their current equilibrium positions to the other. This "wave front" can be assimilated to a dislocation between mismatched sets of devices that regularly alternate opposite equilibrium positions. The emerging spatiotemporal patterns reported in [9] consist in the (random) creation and annihilation of such dislocations. We show that such relaxation process has an infinite average mean life. These features are typical of glassy systems.

II. MODEL

We consider a one-dimensional chain of overdamped two well potential oscillators in which the last is linked to the

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first so to form a ring. Let x_i be the amplitude of the ith oscillator, $i=0,1\ldots,N$, and $V(x_i)$ be a double-well potential with minima of depth U_o located at $x=\pm c$ [1]:

$$V(x) = -U_o\left(\frac{x}{c}\right)^2\left[2-\left(\frac{x}{c}\right)^2\right]. \tag{1}$$

These devices are assumed to be coupled by directional interactions. In what follows we use $U_o=256$ and $c=\sqrt{4U_o/a}$ with $a=32$ measures the curvature of the potential barrier separating both minima. Each device is thus submitted to an external excitation that is proportional to the amplitude of the preceding device in the chain. In addition all devices are assumed to be in contact with uncorrelated sources of Gaussian noise whose variance is represented by a temperaturelike parameter T. The dynamics is therefore given by the set of equations

$$\frac{dx_i}{dt} + P(x_i) = \varepsilon_i x_{i-1} + \xi(t), \tag{2}$$

where

$$P(x_i) = \frac{\partial V(x_i)}{\partial x_i} \tag{3}$$

is a third degree polynomial and $\langle\xi(t)\xi(t')\rangle=2k_BT\delta(t-t')$. The coupling constants ε_i represent attractive ($\varepsilon_i>0$) or repulsive ($\varepsilon_i<0$) connections.

If $\varepsilon_i>0\ \ \forall N$ the system has two obvious equilibrium positions for any value of N in which all devices are in the same minimum of the potential. For the sake of concreteness we refer to both minima, respectively, as "left" (L) or "right" (R). Since this interaction between oscillators is attractive, all devices will tend to act coherently. The N devices will then reach a unique configuration in which all are in the same minimum of the potential.

The situation in which some of these couplings are negative is far more interesting. A negative coupling of the ith device that is presently, say, in the (L) minimum, forces the $(i+1)$th device to be in the (R) minimum. If there is an odd number of negative connections the driving force felt by each device has a sign that prevents it to stabilize in either of its two equilibrium positions. When ε_i and noise have the proper values the system may engage in a solitonlike traveling wave reported in [9] that reminds the (impossible) Escher fountain [10].

When involving a large number of particles, a system as the one considered here can be related to what is known in statistical physics as a *frustrated* systems [11]. In the present case, frustration arises due to the periodic boundary conditions rather than from the interaction as in the case of a spin glass. As we will soon see, in a ring with an odd number of devices and weak enough negative coupling, can be accommodated in any of several [actually $O(N)$] possible different configurations. Any configuration can be characterized through a (zero temperature) quasienergy defined as

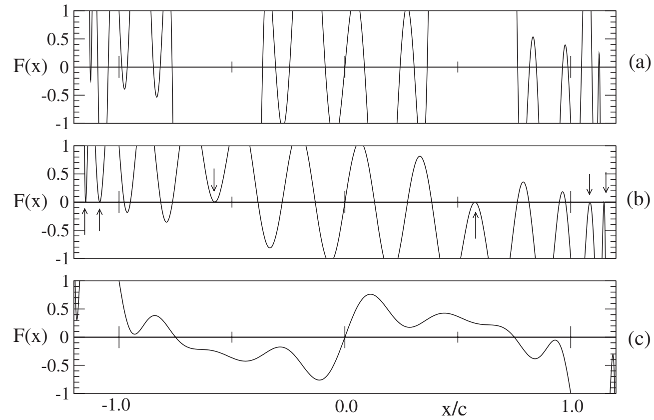

FIG. 1. Plot of the 27th degree polynomial $F(x)$ of Eq. (5) for three different values of α. Panel (a) $|\alpha|>|\alpha_c|$, panel (b) $|\alpha|=|\alpha_c|$ $=2.981\,77$, and panel (c) $|\alpha|<|\alpha_c|$. Note that for smaller values of $|\alpha|$ only survive three roots until $|\alpha|=1$.

$$E = \sum_{i=1}^{i=N}\left[V(x_i) + \varepsilon_i x_i x_{i+1}\right]. \tag{4}$$

With this convention those equivalent configurations have the same (minimal) quasienergy.

III. SYSTEMS WITH FEW DEVICES

We will now concentrate our discussion in the simplest system involving only three bistable oscillators connected by directional negative connections. From the point of view of dynamical systems the points of the phase space associated to N bistable devices can be specified by the N coordinates x_i of the oscillators. The system is at rest in all the points in phase space in which $dx_i/dt=0,\forall i$. Out of all these configurations, only few are stable because in most cases upon small perturbations the equations of motion (2) drive the oscillators to a different state.

If we assume that all constants ε_i have the same (negative) value ε, the static configurations correspond to the roots of the 27th degree algebraic equation

$$\left[\frac{1}{\varepsilon}P\right]^{(3)}(x) - x = 0 \ \text{ with } \ P(x) = ac[(x/c)^3 - (x/c)], \tag{5}$$

where $P^{(N)}(.)$ is the Nth iteration of the function $P(.)$. The locations of all the roots of Eq. (5) change with the value of ε. If ε is weak enough ($|\varepsilon|<|\varepsilon_c|$) Eq. (5) has 27 roots. To see this one can define a control parameter $\alpha=a/\varepsilon$ that is the only parameter defining the roots of Eq. (5). In Fig. 1 we plot the value of the polynomial $F(x)=\frac{1}{c}([\frac{1}{\varepsilon}P]^{(3)}(x)-x)$ for three different values of α. There is a critical value of α for which several pairs of real roots into double roots. This corresponds to the critical value ε_c that for the current values of a turns out to be $-10.731\,88$. We stress that the only potential parameter defining this critical value of the coupling is a, the curvature of the potential barrier at the origin.

The accessible portion of the phase space is restricted to a cube in which the coordinates of the three oscillators are limited to the same interval that corresponds to their maxi-

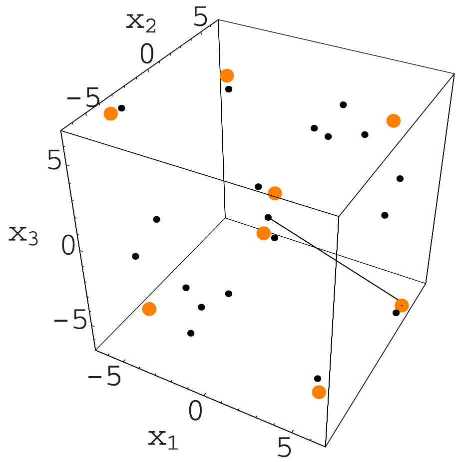

FIG. 2. (Color online) Location of the roots of Eq. (5) for $|\varepsilon| < |\varepsilon_c|$. Big, gray (orange online) dots correspond to static equilibria. Small black dots represent unstable rest positions that except that that of the origin, are saddle points. As an example we show with a line the orbit followed by the system when it is initially located in that point.

mum amplitude. In Fig. 2 we represent with big gray (orange online) dots in this three-dimensional phase space the equilibria that are stable under small perturbations while the remaining unstable roots are represented by small black dots. Six stable rest positions correspond to a node having a closely neighboring saddle point that are related to the pairs of roots indicated above. These are stable along two directions but unstable along the third one. When the interaction becomes stronger than ε_c, the stable and saddle points merge and the system is forced to follow an orbit that goes through positions that are close to the static equilibria of a lower value of the coupling (see Fig. 3). This transition corresponds to a global bifurcation [12] in which several stable roots disappear at the same time and are replaced by a stable, periodic orbit with a period P that approximately follows the edges of the cube of Fig. 2. The three oscillators take

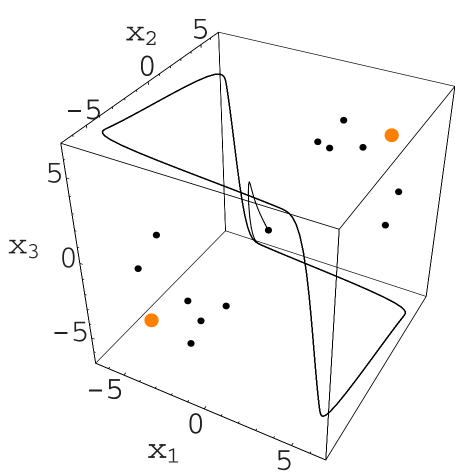

FIG. 3. (Color online) The same as in Fig. 2 but with $|\varepsilon| > |\varepsilon_c|$. The system has been initially located at the unstable point of the origin. Any small perturbation drives the system into the stable, periodic orbit shown in full line.

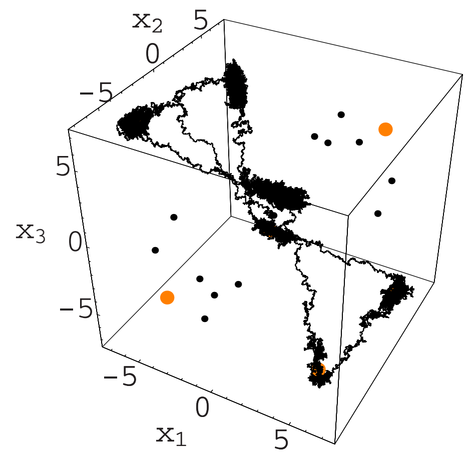

FIG. 4. (Color online) The same as in Fig. 2 but with noise. If initially located at the origin, the system is driven into a noisy orbit that is similar to the one shown in Fig. 3 that is obtained without noise and with a greater negative coupling. Notice that the system tends to stay for a longer time in the neighborhood of the stable equilibria.

successive turns in changing from the (R) to the (L) minimum and can thus be regarded as a traveling wave along the ring. The frequency ($\nu = 2\pi / P$) of this periodic orbit changes with ε.

Besides the above dynamical response of the system, there are two other static equilibrium positions without neighboring saddle points that remain symmetrically located in opposite vertices of the cube. These correspond to roots of $P(x)/\varepsilon = x$ and correspond to having all devices on the same side of the two well potential and survive for rings with any (odd) number of devices. These configurations are always stable until the strength of the coupling becomes $\varepsilon = -a$. However, their basin of attraction becomes negligible for an increasing number of devices.

Let us now consider the system for $|\varepsilon| < |\varepsilon_c|$ i.e., before the global bifurcation has taken place but submitted to external independent sources of random, Gaussian noise in each oscillator. Except for obvious fluctuations, it is seen that the same orbit corresponding to a higher value of $|\varepsilon|$ is recovered. The addition of noise is therefore equivalent to an effective $|\varepsilon| > |\varepsilon_c|$ (see Fig. 4).

The presence of noise increases the probability of jumping across the central barrier of the bistable potential. This feature adds a dependence of ν on the noise amplitude T. External noise therefore allows to recover the same dynamic that a stronger negative interaction would have produced. This is a completely similar situation as the one found in [4] for the case of an externally driven system. The effects of noise that in this latter case may be considered as forcing the system to cross a separatrix, in the case of an autonomous system considered here produces instead a global bifurcation.

IV. SYSTEMS WITH MANY DEVICES

The discussion for a three-dimensional phase space can be generalized to many degrees of freedom provided that the same basic mechanism of frustration is preserved. Most of the features of statistical frustrated system with a large number of degrees of freedom are already displayed in the three-dimensional phase space. The static configurations parallel the many equivalent minima in the free-energy landscape of statistical frustrated systems. In addition static configurations appear to be organized forming basins of attraction. A moderate Gaussian noise does not completely eliminate such basins and the phase space appears to be fragmented in a single periodic orbit plus static configurations.

The underlying mechanism producing the solitonlike waves that we have discussed above can be extended to systems with an increasing number of devices. In what follows we concentrate in the case in which ε is such that a solitonlike wave is present. Under these conditions, the main change that is brought by the presence of many devices is that several traveling waves may coexist. The ring thus appears to be segmented in regions. In each of them all devices alternate in their equilibrium positions and the border between two of these regions consists in a pair of oscillators that temporarily are in the same equilibrium positions. The changes in the oscillator states produce a wave front that travels around the ring.

It is practical to map the dynamical system composed by the ensemble of bistable devices into one in which are only involved traveling dislocations that are composed by pairs of neighboring oscillators that are in similar equilibrium positions say …RLR**LL**RLR…. Within this picture, two colliding dislocations consist in a pattern such as …RLR**LLL**RLR…. Both dislocations may then annihilate each other restoring the chain to an ordered pattern …RLRLRL…. On the other hand, if a thermal fluctuation changes, …RLRLRLRL… → …RLR**LLL**RL…, it can be interpreted as the (thermal) creation of a pair of dislocations.

The coupling between neighboring oscillators also causes that interaction of the dislocations with each other. This interaction depends upon the instantaneous value of the coordinates of close and more distant neighboring oscillators. This, together with the presence of external noise, causes that two interacting dislocations change their traveling speeds getting closer, colliding and annihilating each other. A complex relaxation process therefore takes place by which dislocations may be annihilated but also created by thermal fluctuations. This produces the complex spatiotemporal patterns that have been reported in [9]. The relaxation process is essentially governed by the range of the interaction between dislocations. As a consequence, there exists a critical number N_c that separate two regimes that respectively hold for few and many devices. For $N<N_c$ the number of dislocations that are introduced by the initial conditions rapidly drops until a minimum value of 1 or 0 is reached, respectively, for N odd or even.

The value of N_c can be estimated from the fact that, in order to survive, all dislocations must be equally spaced; therefore, $N=n_d(2+n)$ where n_d represents the number of dislocations ($n_d>1$) and n represents the number of devices between dislocations. Since the minimum value of n is 1, $N_c=3n_d$ relates the critical value of N with the number of dislocations.

For $N \gg N_c$ a short transient leads to configurations in which all the surviving dislocations are nearly equally separated. Since their interaction does not vanish, the distance between them gradually changes and $n_d(t)$ drops at a slower rate, $n_d(t)$ representing here the number of dislocations that survive at time t. In Fig. 5 we show the $n_d(t)$ as a function of the number of devices of the ring, after a large number of time steps ($t\simeq 10^5$). Notice that the system has the surprising property that a moderate temperature favors a greater annihilation of dislocations. The minimum value of $n_d(t)$ is therefore attained for a nonvanishing strength of the noise.

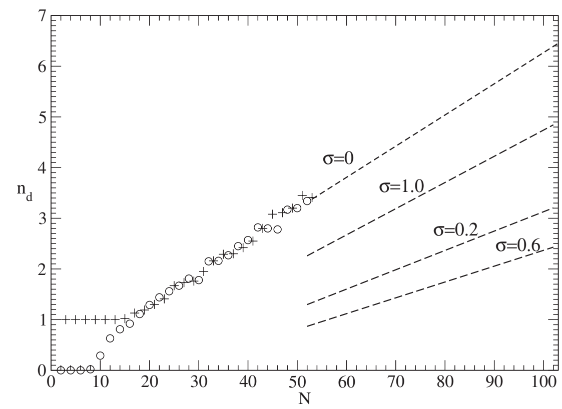

FIG. 5. The number of dislocations $n_d(t)$ surviving after $t \approx 10^5$ iterations is plotted as a function of the number of devices of the ring for different temperatures. Below $N_c \approx 15$ all asymptotic static configurations have been neglected. For clarity scattered symbols are used only for $T=0$; small circles correspond to N odd while crosses correspond to N even. The dashed lines are the linear fit to the results obtained for different temperatures; T is related to the intensity of noise through $\sigma=2k_bT/U_o$. The slope of these lines is the inverse of the average number of devices separating each pair of dislocations. Notice that asymptotically, fewer dislocations are obtained for a nonvanishing temperature.

The tail of the function $n_d(t)$ is shown in Fig. 6 for different temperatures. It is clearly seen that they closely follow a power law $n_d(t) \propto t^{-\alpha}$. Time t is measured in integration steps. As expected, α changes with an increasing intensity of noise. For a very low temperature α becomes small but does not vanish. The fact that $n_d(t)$ is not a constant for $t\to\infty$ is a signature of the range of the interaction between dislocations is of the same order than the size of the system.

The temperature scale has an upper bound that is determined by the Kramers escape time that is in turn associated to the potential barrier separating both minima of the quartic potential. When the temperature exceeds such limit, spontaneous transitions between the two minima become increasingly probable causing the spontaneous creation of pairs of dislocations. On the other hand, as long as the temperature stays below this upper bound, the relaxation process can take place gradually eliminating dislocations. In all cases the exponent of the decay law never exceeds $\alpha=2$ thus producing an infinite average relaxation time.

V. CONCLUSIONS

We have revisited the dynamical system explored in Refs. [8,9] investigating its behavior when both the number of nonlinear devices and the temperature are allowed to change.

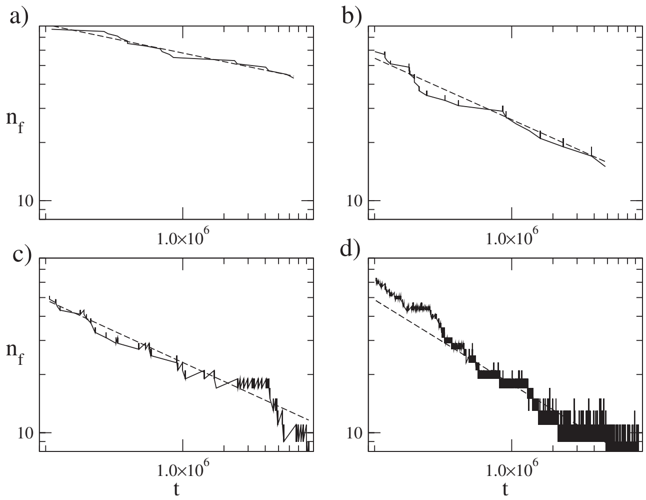

FIG. 6. Value of $n_f(t)$ representing the number of dislocations that survive after a time t. Time is measured in integration steps. Different values of the temperature are shown in different panels. Only the tail of the distribution is shown. The values of the exponent $n_f(t) \propto t^{-\alpha}$ of the best fit of the decay and the intensity of noise $\sigma = 2k_bT/U_o$ in each panel are as follows: (a) $\sigma=0, \alpha=0.147$; (b) $\sigma=0.2, \alpha=0.318$; (c) $\sigma=0.4, \alpha=0.324$; (d) $\sigma=0.6, \alpha=0.437$. Calculations were performed for $N=1003$ devices.

The system that we have considered consists in a chain of bistable devices (quartic oscillators) with directional, effective, repulsive interactions among them that is proportional to the amplitude of the preceding oscillator in the ring. These ingredients produce a frustrated system. For very low values of the interactions the system can only exist at rest in a number of equivalent secondary minima and for larger values it produces a dynamical response to frustration through solitonlike waves. We found that a moderate amount of noise is equivalent to a change in the strength of the coupling between consecutive devices and allows to recover the solitonlike wave.

The transition induced by noise is reminiscent to transitions between heteroclinic orbits [12] in which forces the system into one of two possible equivalent orbits. What is found in this system should instead be understood as the collapse of several saddle-node pairs giving rise to a global bifurcation. Noise therefore drives the system out of rest into a single possible periodic orbit in which all its devices take turns in switching from one minimum of the bistable potential to the other. An additional effect of noise is to change the frequency of the noisy periodic orbit.

The basic ingredients of the solitonlike frustration waves can easily be discussed when the ring involves few bistable devices. When it involves many units the system breaks up in groups of consecutive links that are in opposite minima. Such groups are separated by a pair of oscillators that temporarily live in the same minimum. These are traveling wave fronts and can be regarded as dislocations that interact via long range interactions. The relaxation process that takes place when the system is initialized at random involves complicated spatiotemporal patterns consisting in the collision and annihilation of such dislocations.

We found that this process is essentially governed by the range of the interaction between dislocations. This is related to a critical number of devices that separates the dynamics of rings having few and many links. Below that critical number there is an exponential decay of the number of dislocations. Above, the decay follows a power law with an infinite average mean life. This process also depends upon the intensity of noise. The stationary configurations of a system with an odd (even) number of nonlinear devices are those having an odd (even) number of dislocations that are exactly equally spaced. The infinite mean life of the relaxation process indicates that as the system approaches one of these configurations, it finds higher barriers that turn increasingly improbable to visit configurations with a different symmetry, i.e., with a different (odd or even) number of dislocations. One could thus argue that these systems face some kind of broken ergodicity extending this concept that is only strictly applicable to systems having well defined energy and equilibrium conditions. All these features make this system of coupled bistable devices to behave as a glassy system.

The systems that we have considered here provide another example of the ordering property that can have the addition of moderate noise to an otherwise disordered system. This is so because, given a large number of iteration steps, the minimum number of dislocations and hence the greatest order are attained for a nonvanishing temperature.

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